Some Processes Associated with Fractional Bessel Processes

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Abstract

Let $B = \{(B_t^1, \ldots, B_t^d), t \geq 0\}$ be a d-dimensional fractional Brownian motion with Hurst parameter H and let $R_t = \sqrt{(B_t^1)^2 + \cdots + (B_t^d)^2}$ be the fractional Bessel process. Itô's formula for the fractional Brownian motion leads to the equation $R_t = \sum_{i=1}^d \int_0^t \frac{B_s^i}{R_s} dB_s^i + H(d-1) \int_0^t \frac{s^{2H-1}}{R_s} ds$. In the Brownian motion case ($H = 1/2$), $X_t = \sum_{i=1}^d \int_0^t \frac{B_s^i}{R_s} dB_s^i$ is a Brownian motion. In this paper it is shown that X_t is not a fractional Brownian motion if $H \neq 1/2$. We will study some other properties of this stochastic process as well.

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1 Introduction

Let $B = \{(B_t^1, \ldots, B_t^d), t \geq 0\}$ be a d-dimensional fractional Brownian motion with Hurst parameter $H \in (0,1)$. That is, the components of B are independent one-dimensional fractional Brownian motions with Hurst parameter $H \in (0,1)$.

Denote the fractional Bessel process by $R_t = \sqrt{(B_t^1)^2 + \cdots + (B_t^d)^2}$. In the standard Brownian motion case there is an extensive literature on this process, see for example [7]. It is natural and interesting to study this process for any other parameter H. If $d \geq 2$ and $1/2 < H < 1$, using the Itô's formula for the fractional Brownian motion we obtain

$$R_t = \sum_{i=1}^{d} \int_0^t \frac{B_s^i}{R_s} dB_s^i + H(d-1) \int_0^t \frac{s^{2H-1}}{R_s} ds \tag{1}$$

and for $d = 1$ we have

$$|B_t| = \int_0^t \operatorname{sign}(B_t) dB_t + H \int_0^t \delta_0(B_s) s^{2H-1} ds, \tag{2}$$

where δ_0 is the Dirac delta function, and the stochastic integrals are interpreted in the divergence form. Equation (1) have been proved in [4] in the case $H > \frac{1}{2}$, and for Equation (2) we refer to [1], [4], [5] and [6].

In the classical Brownian motion case it is well-known from the Lévy's characterization theorem that the first term in the decomposition (1)

$$X_t = \begin{cases} \sum_{i=1}^{d} \int_0^t \frac{B_s^i}{R_s} dB_s^i & \text{when} \quad d \geq 2 \\ \\ \int_0^t \operatorname{sign}(B_t) dB_t & \text{when} \quad d = 1 \end{cases}$$

is a classical Brownian motion. It is then natural and interesting to ask whether for any other H, the process $X = \{X_t, t \geq 0\}$ is a fractional Brownian motion or not. The difficulty is that there is no characterization as convenient as Lévy's one for general fractional Brownian motion ($H \neq 1/2$). It is then difficult to show whether a stochastic process is a fractional Brownian motion or not. In this paper, we shall prove that if $H \neq 1/2$, then $\{X_t, t \geq 0\}$ is NOT a fractional Brownian motion. Our approach to show this fact is based on the Wiener chaos expansion (see for example [3] and [5]).

It seems to be the natural method to be used here since there is no other powerful tool available.

Although $\{X_t,\, t \geq 0\}$ is not a fractional Brownian motion, it enjoys some properties that the fractional Brownian motion has, such as self-similarity and long range dependence ($H > 2/3$). We will study these and some other properties of the process X.

Section 2 will recall some preliminary results. Section 3 will study the case $d = 1$, namely, the process $\int_0^t \operatorname{sign}(B_t) dB_t$ and Section 4 is devoted to the study of general dimension, ie, the process $\sum_{i=1}^d \int_0^t \frac{B_s^i}{R_s} dB_s^i$.

2 Preliminaries

Let $B = \{B_t, t \geq 0\}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$. That is, B is a zero mean Gaussian process with the covariance function

$$R_H(t,s) = E(B_t B_s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

We denote by $K_H(t,s)$ the square integrable kernel such that

$$R_H(t,s) = \int_0^{t \wedge s} K_H(t,u) K_H(s,u) du.$$

Fix a time interval $[0,T]$, and let $\mathcal{H}$ be Hilbert space defined as the closure of the set of step functions on $[0,T]$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}} = R_H(t,s).$$

The mapping $\mathbf{1}_{[0,t]} \longrightarrow B_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(B)$ associated with B. We will denote this isometry by $\varphi \longrightarrow B(\varphi)$.

The operator defined by

$$\left(K_H^* \mathbf{1}_{[0,t]}\right)(s) = K_H(t,s) \mathbf{1}_{[0,t]}(s).$$

can be extended to a linear isometry between $\mathcal{H}$ and $L^2(0,T)$. This operator can be expressed in terms of fractional operators. More precisely, if $H > \frac{1}{2}$ we have

$$\left(K_H^* \varphi\right)(s) = c_H \Gamma(H - \frac{1}{2}) s^{\frac{1}{2}-H} (I_{T-}^{H-\frac{1}{2}} u^{H-\frac{1}{2}} \varphi(u))(s)$$

and if $H < \frac{1}{2}$

$$\left(K_H^*\varphi\right)(s) = c_H \Gamma(H + \frac{1}{2})\ s^{\frac{1}{2}-H} (D_{T-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}} \varphi(u))(s),$$

where $c_H = \sqrt{\frac{2H}{(1-2H)\beta(1-2H,H+1/2)}}$, and for any $\alpha > 0$ we denote by I_{T-}^{α} (resp. D_{T-}^{α}) the fractional integral (resp. derivative) operator given by

$$I_{T-}^{\alpha} f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} f(s)\, ds$$

(resp.

$$D_{T-}^{\alpha} f(t) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(T-t)^{\alpha}} + \alpha \int_t^T \frac{f(t) - f(s)}{(s-t)^{\alpha+1}} ds \right) \Bigg).$$

We denote by D and δ the derivative and divergence operators that can be defined in the framework of the Malliavin calculus with respect to the process B. Let $\mathbb{D}^{k,p}$, $p > 1$, $k \in \mathbb{R}$, be the corresponding Sobolev spaces. We recall that the divergence operator δ is defined by means of the duality relationship

$$\mathbb{E}(F\delta(u)) = E \left\langle DF, u \right\rangle_{\mathcal{H}}, \tag{3}$$

where u is a random variable in $L^2(\Omega; \mathcal{H})$. We say that u belongs to the domain of the divergence, denoted by Dom δ, if there is a square integrable random variable $\delta(u)$ such that (3) holds for any $F \in \mathbb{D}^{1,2}$.

The domain of the divergence operator is sometimes too small. For instance, in [1] it is proved that the process $u = B$ belongs to $L^2(\Omega; \mathcal{H})$ if and only if $H > \frac{1}{4}$. On the other hand, in [2] it is proved that for all $t \geq 0$, the process sign(B_t) belongs to the domain of the divergence when $H > \frac{1}{3}$.

Following the approach of [1] it is possible to extend the domain of the divergence operator to processes whose trajectories are not necessarily in the space $\mathcal{H}$. Set $\mathcal{H}_2 = \left(K_H^*\right)^{-1} \left(K_H^{*,a}\right)^{-1} \left(L^2(0,T)\right)$, where $K_H^{*,a}$denotes the adjoint of the operator K_H^*. Denote by $\mathcal{S}_{\mathcal{H}}$ the space of smooth and cylindrical random variables of the form

$$F = f(B(\phi_1), \ldots, B(\phi_n)), \tag{4}$$

where $n \geq 1$, $f \in C_b^{\infty}(\mathbb{R}^n)$ (f and all its partial derivatives are bounded), and $\phi_i \in \mathcal{H}_2$.

Definition 1 *Let* $u = \{u_t, t \in [0,T]\}$ *be a measurable process such that*

$$\mathbb{E}\left(\int_0^T u_t^2 dt\right) < \infty.$$

We say that $u \in \mathrm{Dom}_T^* \delta$ *(extended domain of the divergence in* $[0,T]$*) if there exists a random variable* $\delta(u) \in L^2(\Omega)$ *such that for all* $F \in \mathcal{S}_{\mathcal{H}}$ *we have*

$$\int_0^T \mathbb{E}(u_t K_H^{*,a} K_H^* D_t F) dt = \mathbb{E}(\delta(u) F).$$

In [1] it is proved that for any $H \in (0,1)$, the process $\mathrm{sign}(B_t)$ belongs to the extended domain of the divergence in any time interval $[0,T]$ and the following version of Tanaka's formula holds

$$|B_t| = \int_0^t \mathrm{sign}(B_s) dB_s + H \int_0^t \delta_0(B_s) s^{2H-1} ds. \tag{5}$$

(see also [5], [6] for this and a more general formula). In this formula $L_t^a = H \int_0^t \delta_0(B_s) s^{2H-1} ds$ is the the density of the occupation measure

$$\Gamma \mapsto 2H \int_0^t 1_\Gamma(B_s) s^{2H-1} ds.$$

3 The process $\int_0^t \mathrm{sign}(B_t) dB_t$

Define the process $X = \{X_t, t \geq 0\}$, by

$$X_t = \int_0^t \mathrm{sign}(B_s) dB_s.$$

In the case of the classical Brownian motion ($H = \frac{1}{2}$), the process X turns out to be a Brownian motion. We will show first that for any $H \in (0,1)$, X is a H-self-similar process, that is, for all $a > 0$ the processes $\{X_{at}, t \geq 0\}$ and $\{a^H X_t, t \geq 0\}$ have the same law.

Proposition 2 *The process* $X = \{X_t, t \geq 0\}$ *is* H*-self-similar.*

Proof. Using the self-similarity property of the fractional Brownian motion and Tanaka's formula (5) yields that for any $a > 0$

$$\begin{aligned}
X_{at} &= |B_{at}| - H\int_0^{at} \delta_0(B_s)s^{2H-1}ds \\
&= |B_{at}| - H\int_0^{t} \delta_0(B_{au})(au)^{2H-1}adu \\
&\stackrel{d}{=} a^H|B_t| - a^{2H}H\int_0^{t} \delta_0(a^H B_u)u^{2H-1}du \\
&= a^H X_t,
\end{aligned}$$

where the symbol $\stackrel{d}{=}$ means that the distributions of both processes are the same. This completes the proof. ■

Then, it is natural to conjecture that for any H, the process X_t is a fractional Brownian motion of Hurst parameter H. We will see that this is no longer true if $H \neq \frac{1}{2}$, although the process X_t shares some of the properties of the fractional Brownian motion.

Let us first find the Wiener chaos expansion of the process $\mathrm{sign}(B_t)$. We will denote by I_n the multiple Wiener integral with respect to the process B.

Lemma 3 *Let $0 < H < 1$. We have the following chaos expansion for* $\mathrm{sign}(B_t)$*:*

$$\mathrm{sign}(B_t) = \sum_{k=0}^{\infty} b_{2k+1} I_{2k+1}(1), \tag{6}$$

where

$$b_{2k+1} = \frac{2(-1)^k}{(2k+1)\sqrt{2\pi}t^{(2k+1)H}k!2^k}.$$

Proof. Denote by $p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}}e^{-x^2/\varepsilon}$, $x \in \mathbb{R}$, $\varepsilon > 0$, the heat kernel. The function

$$f_\varepsilon(x) = 2\int_{-\infty}^{x} p_\varepsilon(y)dy - 1$$

converges to $\mathrm{sign}(x)$ as ε tends to zero. Hence, $f_\varepsilon(B_t)$ converges to $\mathrm{sign}(B_t)$ in $L^2(\Omega)$ as ε tends to zero. The application of Stroock's formula yields

$$f_\varepsilon(B_t) = \sum_{n=0}^{\infty} a_n^\varepsilon(t) \int_{0<s_1<\cdots<s_n<t} dB_{s_1}\cdots dB_{s_n}, \tag{7}$$

where

$$
\begin{aligned}
a_n^\varepsilon(t) &= \mathbb{E}\left[D^n(f_\varepsilon(B_t))\right] = 2\mathbb{E}\left[p_\varepsilon^{(n-1)}(B_t)\right] \\
&= 2(-1)^{n-1}\frac{\partial^{n-1}}{\partial y^{n-1}}\mathbb{E}\left[p_\varepsilon(B_t-y)\right]|_{y=0} \\
&= 2(-1)^{n-1}p_{\varepsilon+t^{2H}}^{(n-1)}(0).
\end{aligned}
$$

Taking the limit of (7) in $L^2(\Omega)$ as ε tends to zero we obtain

$$
\operatorname{sign}(B_t) = \sum_{n=0}^{\infty} a_n(t) \int_{0<s_1<\cdots<s_n<t} dB_{s_1}\cdots dB_{s_n},
$$

where $a_n(t) = \lim_{\varepsilon\downarrow 0} a_n^\varepsilon(t) = 2(-1)^{n-1}p_{t^{2H}}^{(n-1)}(0)$. As a consequence, $a_n(t) = 0$ if n is even and

$$
a_n(t) = \frac{2(-1)^k(2k)!}{\sqrt{2\pi}t^{nH}k!2^k}
$$

if $n = 2k+1$. ■

Using Stirling's formula we obtain

$$
\begin{aligned}
\mathbb{E}\left[I_{2k+1}(b_{2k+1})\right]^2 &= \frac{4(2k+1)!t^{2H+1}}{(2k+1)2\pi t^{2H+1}\left(k!2^k\right)^2} \\
&= \frac{4(2k)!}{(2k+1)2\pi\left(k!2^k\right)^2} \\
&\backsimeq Ck^{-3/2},
\end{aligned}
$$

and we have proved the following proposition.

Proposition 4 *For any $0 < H < 1$, the random variable* $\operatorname{sign}(B_t)$ *belongs to the Sobolev space $\mathbb{D}^{\alpha,2}$ for any $\alpha < \frac{1}{2}$.*

Now it is easy to obtain the chaos expansion of $\int_0^t \operatorname{sign}(B_s)dB_s$.

Proposition 5 *For any $0 < H < 1$,*

$$
\int_0^t \operatorname{sign}(B_s)dB_s = \sum_{k=1}^{\infty} c_k I_{2k}(h_{2k}),
$$

where

$$c_k = \frac{(-1)^{k-1}}{\sqrt{2\pi}(2k-1)(k-1)!2^{k-2}}$$

and

$$h_{2k}(s_1, \dots, s_{2k}) = (s_1 \vee s_2 \vee \cdots \vee s_{2k})^{-(2k-1)H} \; .$$

A consequence of this proposition is the following

Proposition 6 *For any $0 < H < 1$ and $t > 0$, the random variable $\int_0^t \operatorname{sign}(B_s)dB_s$ belongs to the Sobolev space $\mathbb{D}^{\alpha,2}$ for any $\alpha < 1/2$.*

Proof. It is easy to check that there is a constant $C > 0$ such that

$$\mathbb{E}\left[I_{2k}(h_{2k})\right]^2 \leq C \frac{(2k)!}{(2k-1)^2\left[(k-1)!\right]^2 2^{2k}} \, .$$

Therefore

$$\begin{aligned} \mathbb{E}\left[c_k I_{2k}(h_{2k})\right]^2 &\leq C \frac{(2k)!}{(k!2^k)^2} \\ &\leq Ck^{-3/2} \, . \end{aligned}$$

This proves the proposition. ■

The next proposition states that $\int_0^t \operatorname{sign}(B_t)dB_t$ is not a fractional Brownian motion.

Proposition 7 *The process $X = \{X_t, t \geq 0\}$ is not a fractional Brownian motion.*

Proof. Suppose that X is a fractional Brownian motion. Then it is a fractional Brownian motion with Hurst parameter H since it is self-similar with parameter H. Then, the process

$$Y_t = \int_0^t \eta_H(t,r)dX_r$$

must be a standard Brownian motion with respect to the filtration generated by X, where

$$\eta_H(t,r) = (K_H^*)^{-1}(\mathbf{1}_{[0,t]})(r).$$

We claim that

$$Y_t = Z_t, \tag{8}$$

where

$$Z_t = \int_0^t \eta_H(t,r)\mathrm{sign}(B_r)dB_r. \tag{9}$$

In fact, set $t_k^n = \frac{tk}{n}$, $k = 0, \dots, n$, and consider the approximations

$$Y_t^n = \sum_{k=1}^n \eta_H(t, t_{k-1}^n)\left(X_{t_k^n} - X_{t_{k-1}^n}\right).$$

We know that Y_t^n converges in $L^2(\Omega)$ to Y_t, because the functions

$$\sum_{k=1}^n \eta_H(t, t_{k-1}^n)\mathbf{1}_{[t_{k-1}^n, t_k^n)}(r)$$

converge to $\eta_H(t,r)\mathbf{1}_{[0,t)}(r)$ in the norm of the Hilbert space $\mathcal{H}$. On the other hand, by Definition 1, for any smooth and cylindrical random variable $F \in \mathcal{S}_{\mathcal{H}}$ we have

$$\begin{aligned}
\mathbb{E}(FY_t^n) &= \mathbb{E}\left(\left\langle D_r F, \sum_{k=1}^n \eta_H(t, t_{k-1}^n)\mathbf{1}_{[t_{k-1}^n, t_k^n)}(r)\mathrm{sign}(B_r)\right\rangle_{\mathcal{H}}\right) \\
&= \mathbb{E}\left(\left\langle \Gamma_{H,T}^{*,a}\Gamma_{H,T}^{*} D_r F, \sum_{k=1}^n \eta_H(t, t_{k-1}^n)\mathbf{1}_{[t_{k-1}^n, t_k^n)}(r)\mathrm{sign}(B_r)\right\rangle_{L^2(0,T)}\right).
\end{aligned}$$

As before this converges to

$$\begin{aligned}
&\mathbb{E}\left(\left\langle K^{*,a}K^{*}D_r F, \eta_H(t,r)\mathbf{1}_{[0,t)}(r)\mathrm{sign}(B_r)\right\rangle_{L^2(0,T)}\right) \\
=\ &\mathbb{E}(FZ_t),
\end{aligned}$$

as n tends to infinity. So (8) holds.

We can write, using Lemma 3

$$Z_t = \sum_{k=0}^\infty b_k \int_0^t \eta_H(t,r) r^{-(2k+1)H} I_{2k+1}(\mathbf{1}_{[0,r]}^{\otimes(2k+1)})dB_r,$$

where

$$b_k = \frac{(-1)^k}{\sqrt{2\pi}(2k+1)k!2^{k-1}}. \tag{10}$$

So,

$$Z_t = \sum_{k=0}^{\infty} b_k I_{2k+2,t}(f_{2k+2}),$$

where

$$\begin{aligned}
& f_{2k+2}(t, s_1, \ldots, s_{2k+2}) \\
= \; & \operatorname{symm}\left(\eta_H(t, s_{2k+2}) s_{2k+2}^{-(2k+1)H} \mathbf{1}_{[0,2k+2]}(s_1) \cdots \mathbf{1}_{[0,2k+2]}(s_{2k+1})\right) \\
= \; & \frac{1}{2k+1} \eta_H(t, s_1 \vee \cdots \vee s_{2k+2}) \left(s_1 \vee \cdots \vee s_{2k+2}\right)^{-(2k+1)H},
\end{aligned}$$

and $I_{2k+2,t}(f)$ denotes $I_{2k+2}(f \mathbf{1}_{[0,t]}^{\otimes(2k+2)})$. We can transform these multiple stochastic integrals into integrals with respect to a standard Brownian motion, using the operator K_H^*. In this way we obtain

$$I_{2k+2,t}(f_{2k+2}) = I_{2k+2,t}^W(K_H^{*\otimes(2k+2)} f_{2k+2}),$$

and the process

$$Z_t = \sum_{k=0}^{\infty} b_k I_{2k+2,t}^W(K_H^{*\otimes(2k+2)} f_{2k+2})$$

is a Brownian motion with respect to the filtration generated by W. Hence, every component of the chaos expansion is a martingale with respect to the filtration generated by W. In particular, this implies that the coefficient of the second chaos $K_H^{*\otimes 2} f_2(t, s_1, s_2)$ must not depend on t.

For $H > \frac{1}{2}$ we have

$$\begin{aligned}
K_H^{*\otimes 2} f_2(t, s_1, s_2) \; &= \; d_H^2 \; (s_1 s_2)^{\frac{1}{2}-H} \\
& \quad \times \left[I_{t-}^{(H-\frac{1}{2})\otimes 2} (u_1 u_2)^{-\frac{1}{2}} \eta_H(t, u_1 \vee u_2) \right](s_1, s_2)
\end{aligned}$$

where $d_H = c_H \Gamma(H - \frac{1}{2})$. We have used the fact that

$$\left(I_{t-}^{\alpha} f\right) \mathbf{1}_{[0,t]} = I_{T-}^{\alpha}\left(f \mathbf{1}_{[0,t]}\right).$$

Then

$$\begin{aligned}
& \left[I_{t-}^{(H-\frac{1}{2})\otimes 2} (u_1 u_2)^{-\frac{1}{2}} \eta_H(t, u_1 \vee u_2) \right](s_1, s_2) \\
= \; & \frac{1}{\Gamma(H - \frac{1}{2})^2} \int_{s_2}^{t} \int_{s_1}^{t} \left((u_1 - s_1)(u_2 - s_2)\right)^{-\frac{1}{2}} \\
& \times \eta_H(t, (u_1 - s_1) \vee (u_2 - s_2)) du_1 du_2.
\end{aligned}$$

Taking $t = \max(s_1, s_2)$, we would have $K_H^{*\otimes 2} f_2(t, s_1, s_2) = 0$, because

$$\eta_H(t,r) \leq C t^{H-\frac{1}{2}} r^{H-\frac{1}{2}} (t-r)^{\frac{1}{2}-H}.$$

Hence, $\eta_H(t, u_1 \vee u_2) = 0$, which leads to a contradiction.

Suppose now that $H < \frac{1}{2}$. In this case we have

$$\begin{aligned} K_H^{*\otimes 2} f_2(t, s_1, s_2) &= e_H^2 \, (s_1 s_2)^{\frac{1}{2}-H} \\ &\quad \times \left[D_{t-}^{\left(\frac{1}{2}-H\right)\otimes 2} (u_1 u_2)^{-\frac{1}{2}} \eta_H(t, u_1 \vee u_2) \right] (s_1, s_2), \end{aligned}$$

where $e_H = c_H \Gamma(H + \frac{1}{2})$, using again that $\left(D_{t-}^{\alpha} f\right) \mathbf{1}_{[0,t]} = D_{T-}^{\alpha} \left(f \mathbf{1}_{[0,t]}\right)$. Notice that

$$\eta_H(t,r) = \frac{1}{e_H \Gamma(\frac{1}{2} - H)} r^{\frac{1}{2}-H} \int_r^t (y-r)^{-\frac{1}{2}-H} y^{H-\frac{1}{2}} dy.$$

As a consequence, $\eta_H(t,r)$ behaves as $C r^{\frac{1}{2}-H} (t-r)^{\frac{1}{2}-H}$.We have

$$\begin{aligned} &\left[D_{t-}^{\left(\frac{1}{2}-H\right)\otimes 2} (u_1 u_2)^{-\frac{1}{2}} \eta_H(t, u_1 \vee u_2) \right] (s_1, s_2) \\ = \; & \frac{1}{\Gamma\left(H + \frac{1}{2}\right)^2} D_{t-}^{\frac{1}{2}-H} \Bigg(\frac{(s_1 s_2)^{-\frac{1}{2}} \eta_H(t, s_1 \vee s_2)}{(t-s_1)^{\frac{1}{2}-H} (t-s_2)^{\frac{1}{2}-H}} \\ & + \left(\frac{1}{2} - H\right) \int_{s_1}^t \frac{(s_1 s_2)^{-\frac{1}{2}} \eta_H(t, s_1 \vee s_2) - (y s_2)^{-\frac{1}{2}} \eta_H(t, y \vee s_2)}{(y-s_1)^{\frac{3}{2}-H} (t-s_2)^{\frac{1}{2}-H}} dy \\ & + \left(\frac{1}{2} - H\right) \int_{s_2}^t \frac{(s_1 s_2)^{-\frac{1}{2}} \eta_H(t, s_1 \vee s_2) - (y s_1)^{-\frac{1}{2}} \eta_H(t, y \vee s_1)}{(y-s_2)^{\frac{3}{2}-H} (t-s_1)^{\frac{1}{2}-H}} dy \\ & + \left(\frac{1}{2} - H\right)^2 \int_{s_1}^t \int_{s_2}^t (y-s_1)^{H-\frac{3}{2}} (z-s_2)^{H-\frac{3}{2}} (s_1 s_2)^{-\frac{1}{2}} \big[\eta_H(t, s_1 \vee s_2) \\ & - (y s_2)^{-\frac{1}{2}} \eta_H(t, y \vee s_2) - (z s_1)^{-\frac{1}{2}} \eta_H(t, z \vee s_1) + (yz)^{-\frac{1}{2}} \eta_H(t, y \vee z) \big] \, dy dz \Bigg). \end{aligned}$$

Taking again $t = \max(s_1, s_2)$, we would have $K_H^{*\otimes 2} f_2(t, s_1, s_2) = 0$ which leads to a contradiction. ■

Consider the covariance between two increments of the process X:

$$r(n) := \mathbb{E}\left[(X_{a+1} - X_a)(X_{n+1} - X_n) \right],$$

where $0 < a \leq n$. We say that X is long-range dependent if for any $a > 0$,

$$\sum_{n \geq a} |r(n)| = \infty.$$

The next proposition studies the long-range dependence properties of the process X. We see that this property differs from that of fractional Brownian motion.

Proposition 8 *Let $X_t = \int_0^t \operatorname{sign}(B_s) dB_s$. If $H \geq 2/3$, then X_t is long-range dependent and if $1/2 < H < 2/3$, then X_t is not long-range dependent.*

Proof. From Lemma 3 we can deduce the Wiener chaos expansion of the random variable X_t. In fact, we have

$$X_t = \sum_{k=0}^{\infty} b_k I_{2k+2,t}(h_{2k+2}),$$

where b_k is defined in (10) and

$$h_{2k+2}(s_1, \ldots, s_{2k+2}) = (s_1 \vee \cdots \vee s_{2k+2})^{-(2k+1)H} .$$

Let us compute the covariance of X_s and $X_t - X_r$, where $0 < r < t$. From the Itô isometry of multiple stochastic integrals it follows that

$$\mathbb{E}\left[X_s(X_t - X_r)\right] = \sum_{k=0}^{\infty} b_k^2 \mathbb{E}\left[I_{2k+2,s}(h_{2k+2})\left[I_{2k+2,t}(h_{2k+2}) - I_{2k+2,r}(h_{2k+2})\right]\right].$$

We have

$$\begin{aligned}
\mathbb{E}\left[X_s(X_t - X_r)\right] &= \sum_{k=0}^{\infty} b_k^2 (2k+2)! \left\langle h_{2k+2} \mathbf{1}_{[0,s]}^{\otimes(k+2)}, h_{2k+2}\left(\mathbf{1}_{[0,t]}^{\otimes(k+2)} - \mathbf{1}_{[0,r]}^{\otimes(k+2)}\right)\right\rangle_{\mathcal{H}^{2k+2}} \\
&\geq 2b_0^2 \left\langle h_2 \mathbf{1}_{[0,s]}^{\otimes 2}, h_2\left(\mathbf{1}_{[0,t]}^{\otimes 2} - \mathbf{1}_{[0,r]}^{\otimes 2}\right)\right\rangle_{\mathcal{H}^2} \\
&\geq \frac{b_0^2}{2} \int_r^t \int_0^r \int_0^s \int_0^s s_2^{-H} t_2^{-H} \phi(s_1, t_1)\phi(s_2, t_2) ds_1 ds_2 dt_1 dt_2,
\end{aligned}$$

where $\phi(s,t) = H(2H-1)|t-s|^{2H-2}$.

Thus let $s = 1$, $r = n$, and $t = n + 1$ and we have

$$\begin{aligned} r(n) &:= \mathbb{E}\left[(X_{a+1} - X_a)(X_{n+1} - X_n)\right] \\ &\geq C \int_n^{n+1} \int_0^n \int_a^{a+1} \int_a^{a+1} s_2^{-H} t_2^{-H} (t_1 - s_1)^{2H-2} (t_2 - s_2)^{2H-2} ds_1 ds_2 dt_1 dt_2 \\ &\geq C \int_n^{n+1} \int_{a+2}^n \int_a^{a+1} \int_a^{a+1} s_2^{-H} t_2^{-H} (t_1 - a - 1)^{2H-2} (t_2 - a - 1)^{2H-2} ds_1 ds_2 dt_1 dt_2 \\ &\approx Cn^{3H-3}, \end{aligned}$$

as n tends to infinity. Thus if $H \geq 2/3$, $\sum_{n \geq a} r(n) = \infty$.

If $H < 2/3$, then we use another approach. Set, as before

$$r(n) = \mathbb{E}\left[\left(\int_a^{a+1} \operatorname{sign} B_t dB_t\right)\left(\int_n^{n+1} \operatorname{sign} B_t dB_t\right)\right].$$

Using the formula for the expectation of the product of two divergence integrals we obtain

$$\begin{aligned} r(n) &= \alpha_H \int_a^{a+1} \int_n^{n+1} \mathbb{E}\left(\operatorname{sign} B_s \operatorname{sign} B_t\right) \, |s - t|^{2H-2} ds dt \\ &\quad + 4\alpha_H^2 \int_a^{a+1} \int_n^{n+1} \int_0^s \int_0^t \mathbb{E}\left(\delta_0(B_s)\delta_0(B_t)\right) \\ &\quad \times \, |s - \sigma|^{2H-2} |\theta - t|^{2H-2} | d\sigma d\theta ds dt, \end{aligned}$$

where $\alpha_H = H(2H - 1)$. This formula can be proved by approximating the function $\operatorname{sign}(x)$ by smooth functions and then taking the limit in $L^2(\Omega)$. We have

$$\int_0^t |s - \sigma|^{2H-2} d\sigma = \frac{1}{2H - 1}(s^{2H-1} + |s - t|^{2H-1} \operatorname{sign}(t - s)).$$

Hence,

$$\begin{aligned} r(n) &= \alpha_H \int_a^{a+1} \int_n^{n+1} \mathbb{E}\left(\operatorname{sign} B_s \operatorname{sign} B_t\right) \, |s - t|^{2H-2} ds dt \\ &\quad + 4H^2 \int_a^{a+1} \int_n^{n+1} \mathbb{E}\left(\delta_0(B_s)\delta_0(B_t)\right) \\ &\quad \times (s^{2H-1} + (t - s)^{2H-1})(t^{2H-1} - (t - s)^{2H-1}) ds dt \\ &= a_n + b_n. \end{aligned}$$

For the second term we have

$$b_n = \frac{4H^2}{2\pi}\int_a^{a+1}\int_n^{n+1}\frac{(s^{2H-1}+(t-s)^{2H-1})(t^{2H-1}-(t-s)^{2H-1})}{\left[(st)^{2H}-\frac{1}{4}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right)^2\right]^{1/2}}dsdt$$

Therefore,

$$\begin{aligned} b_n &\leq \frac{4H^2(2H-1)}{2\pi}\frac{\left((a+1)^{2H-1}+(n+1)^{2H-1}\right)(n-a-1)^{2H-2}}{\left[(an)^{2H}-\frac{1}{4}\left((n+1)^{2H}+(a+1)^{2H}-|n-a|^{2H}\right)^2\right]^{1/2}} \\ &\leq Cn^{3H-3}. \end{aligned}$$

To estimate a_n, we have from (6)

$$\begin{aligned} \mathbb{E}\left[\operatorname{sign}(B_u)\operatorname{sign}(B_v)\right] &= \sum_{k=0}^{\infty}\frac{4(2k)!}{(2k+1)^2 2\pi(k!2^k)^2(uv)^{(2k+1)H}}(u^{2H}+v^{2H}-|u-v|^{2H})^{2k+1} \\ &\leq C\frac{u^{2H}+v^{2H}-|u-v|^{2H}}{(uv)^H}\sum_{k=0}^{\infty}k^{-3/2}\frac{(u^{2H}+v^{2H}-|u-v|^{2H})^{2k}}{(uv)^{2kH}} \\ &\leq C_1\frac{u^{2H}+v^{2H}-|u-v|^{2H}}{(uv)^H}. \end{aligned}$$

Therefore

$$\begin{aligned} a_n &\leq C_2\int_a^{a+1}\int_n^{n+1}|u-v|^{2H-2}\frac{u^{2H}+v^{2H}-|u-v|^{2H}}{(uv)^H}dvdu \\ &\leq C_3 n^{3H-3}. \end{aligned}$$

As a consequence, if $H < 2/3$, then $\sum_{n\geq 1} r(n) < \infty$. ■

4 General Dimension

In this section we consider a d-dimensional fractional Brownian motion $B = \{(B_t^1,\ldots,B_t^d), t\geq 0\}$, with Hurst parameter $H > \frac{1}{2}$. Let $R_t = |B_t|$ be the fractional Bessel process associated to the d-dimensional fBm B.

Suppose first that $H > \frac{1}{2}$. Fix a time interval $[0,T]$, and define the derivative and divergence operators, $D^{(i)}$ and $\delta^{(i)}$, with respect to each component

$B^{(i)}$, as in Section 2. We assume that the Sobolev spaces $\mathbb{D}_i^{1,p}$ include functionals of all the components of B and not only of component i. For each $p>1$, let $\mathbb{L}_{H,i}^{1,p}$ be the set of processes $u \in \mathbb{D}_i^{1,p}(\mathcal{H})$ such that

$$\mathbb{E}\left[\|u\|_{L^{1/H}([0,T])}^p\right] + \mathbb{E}\left[\left\|D^{(i)}u\right\|_{L^{1/H}\left([0,T]^2\right)}^p\right] < \infty.$$

It has been proved in [1] that $\left\{\frac{B_s^i}{R_s}, s \in [0,T]\right\}$ belongs to the space $\mathbb{L}_{H,i}^{1,1/H}$ for each $i = 1, \ldots, d$ and

$$R_t = \sum_{i=1}^d \int_0^t \frac{B_s^i}{R_s} dB_s^i + H(d-1) \int_0^t \frac{s^{2H-1}}{R_s} ds. \tag{11}$$

In the case $H < \frac{1}{2}$ the following result holds.

Proposition 9 *If $H < \frac{1}{2}$, the process $\frac{B_s^i}{R_s}$ belongs to the extended domain of the divergence operator $\mathrm{Dom}_t^* \delta^i$ on any time interval $[0,t]$, and (11) holds.*

Proof. For any test random variable $F \in \mathcal{S}_{\mathcal{H}}$ we have

$$\begin{aligned}
& \int_0^t \mathbb{E}\left(\frac{B_s^i}{R_s} K_H^{*,a} K_H^* D_s^{(i)} F\right) ds \\
= & \lim_{\varepsilon \downarrow 0} \int_0^t \mathbb{E}(h_\varepsilon'(R_s^2) B_s^i) K_H^{*,a} K_H^* D_s^{(i)} F) ds \\
= & \lim_{\varepsilon \downarrow 0} \mathbb{E}\left(F \int_0^t h_\varepsilon'(R_s^2) B_s^i dB_s^i\right),
\end{aligned}$$

where, for any $\varepsilon > 0$,

$$h_\varepsilon(x) = \begin{cases} \frac{3}{8}\sqrt{\varepsilon} + \frac{3}{4\sqrt{\varepsilon}} x - \frac{1}{8\varepsilon\sqrt{\varepsilon}} x^2 & \text{if } x < \varepsilon \\ \sqrt{x} & \text{if } x \geq \varepsilon \end{cases}.$$

We have $h_\varepsilon(x) \in C^2(\mathbb{R})$ and $\lim_{\varepsilon \to 0} h_\varepsilon(x) = \sqrt{x}$ for all $x \geq 0$. By Itô's formula for the fractional Brownian motion in the case $H < \frac{1}{2}$, we have

$$h_\varepsilon(R_t^2) - h_\varepsilon(0) = \sum_{i=1}^d \int_0^t h_\varepsilon'(R_s^2) B_s^i dB_s^i + J_\varepsilon,$$

where

$$\begin{aligned} J_\varepsilon &= H(d-1)\int_0^t \mathbf{1}_{\{R_s^2\geq\varepsilon\}}\frac{s^{2H-1}}{R_s}ds \\ &\quad +H\int_0^t \mathbf{1}_{\{R_s^2<\varepsilon\}}\frac{1}{2\sqrt{\varepsilon}}\left[3d-(d+2)\frac{R_s^2}{\varepsilon}\right]s^{2H-1}ds. \end{aligned}$$

■

The process

$$X_t = \sum_{i=1}^d \int_0^t \frac{B_s^i}{R_s}\delta B_s^i \tag{12}$$

is H-self-similar, Hölder continuous of order $\alpha < H$, and it has the same $\frac{1}{H}$-variation as the fractional Brownian motion. Nevertheless, as we will show in the next proposition it is not a fractional Brownian motion with Hurst parameter H.

For $h \in \mathcal{H}^{\otimes n}$, we denote

$$I_{j_1,\dots,j_n}(h) = \int_{0<s_1,\dots,s_n<t} h(s_1,\dots,s_n)dB_{s_1}^{j_1}\cdots dB_{s_n}^{j_n}, \tag{13}$$

First we find the chaos expansion of $\sum_{i=1}^d \int_0^t f_i(B_s)dB_s^i$, where $f_i:\mathbb{R}^d\to\mathbb{R}$ are smooth functions with polynomial growth.

Proposition 10 *The following chaos expansion holds for* $Z_t = \sum_{i=1}^d \int_0^t f_i(B_s)dB_s^i$

$$Z_t = \sum_{i=1}^d\sum_{n=1}^\infty \sum_{1\leq j_1,\dots,j_n\leq d} I_{j_1,\dots,j_n,i}\left(g^i_{j_1,\dots,j_n}(s_1,\dots,s_{n+1}\right),$$

where

$$\begin{aligned} g^i_{j_1,\dots,j_n}(s_1,\dots,s_{n+1}) &= \frac{(-1)^n(s_1\vee\cdots\vee s_{n+1})^{-nH}}{(2\pi)^{d/2}} \\ &\quad \times\int_{\mathbb{R}^d}\left[\frac{\partial^n}{\partial y_{j_1}\cdots\partial y_{j_n}}e^{-\frac{|y|^2}{2}}\right]f_i(y(s_1\vee\cdots\vee s_{n+1})^H)dy. \end{aligned}$$

Proof. Using Stroock's formula yields for each $i=1,\dots,d$

$$f_i(B_s) = \sum_{n=0}^\infty \sum_{1\leq j_1,\dots,j_n\leq d}\frac{1}{n!}I_{j_1,\dots,j_n}\left(f^i_{j_1,\dots,j_n}(s)\mathbf{1}_{[0,s]}^{\otimes n}\right),$$

where

$$
\begin{aligned}
f^i_{j_1,\dots,j_n}(s) &= \mathbb{E}(D^{j_1}\cdots D^{j_n}(f_i(B_s))) \\
&= \mathbb{E}(\frac{\partial^n f_i}{\partial z_{j_1}\cdots\partial z_{j_n}}(B_s)) \\
&= \frac{1}{(2\pi s^{2H})^{d/2}}\int_{\mathbb{R}^d}\frac{\partial^n f_i}{\partial z_{j_1}\cdots\partial z_{j_n}}(z)e^{-\frac{|z|^2}{2s^{2H}}}dz \\
&= \frac{s^{-nH}}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\frac{\partial^n f_i}{\partial y_{j_1}\cdots\partial y_{j_n}}(ys^H)e^{-\frac{|y|^2}{2y}}dy \\
&= \frac{(-1)^n s^{-nH}}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f_i(ys^H)\frac{\partial^n}{\partial y_{j_1}\cdots\partial y_{j_n}}e^{-\frac{|y|^2}{2y}}dy.
\end{aligned}
$$

Finally, the result follows from

$$
\begin{aligned}
Z_t &= \sum_{i=1}^d \int_0^t f_i(B_s)dB^i_s \\
&= \sum_{i=1}^d\sum_{n=0}^\infty\sum_{1\le j_1,\dots,j_n\le d} I_{j_1,\dots,j_n,i}\left(\text{symm}\left(f^i_{j_1,\dots,j_n}(s)\mathbf{1}^{\otimes n}_{[0,s]}\mathbf{1}_{[0,t]}(s)\right)\right) \\
&= \sum_{i=1}^d\sum_{n=0}^\infty\sum_{1\le j_1,\dots,j_n\le d} I_{j_1,\dots,j_n,i}\left(f^i_{j_1,\dots,j_n}(s_1\vee\cdots\vee s_{n+1})\prod_{i=1}^{n+1}\mathbf{1}_{[0,t]}(s_i)\right),
\end{aligned}
$$

which completes the proof of the proposition. ■

Now let $f_i(x) = \frac{x_i}{\sqrt{x_1^1+\cdots+x_d^2}}$. Then it is easy to check $f_i(tx) = f_i(x)$ for all $t > 0$. Hence, for such f_i, we have

$$
g^i_{j_1,\dots,j_n}(s_1,\dots,s_{n+1}) = b_{j_1,\cdots,j_n}(s_1\vee\cdots\vee s_{n+1})^{-nH},
$$

where

$$
b_{i,j_1,\dots,j_n} = \frac{(-1)^n}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}\left[\frac{\partial^n}{\partial y_{j_1}\cdots\partial y_{j_n}}e^{-\frac{|y|^2}{2}}\right]f_i(y)dy.
$$

Then the chaos expansion of $f_i(B_t)$ is given by

$$
f_i(B_t) = \sum_{n=0}^\infty\sum_{1\le j_1,\dots,j_n\le d} b_{i,j_1,\dots,j_n}t^{-nH}\int_{\{0<s_1<\cdots<s_n<t\}} dB^{j_1}_{s_1}\cdots dB^{j_n}_{s_n},
$$

and the chaos expansion of the divergence of this process is

$$\begin{aligned}\int_0^t f_i(B_s)dB_s^i &= \sum_{n=1}^{\infty}\sum_{1\le j_1,\dots,j_n\le d} b_{i,j_1,\dots,j_n} \\ &\times \int_{\{0<s_1,\dots,s_{n+1}<t\}} (s_1\vee\cdots\vee s_{n+1})^{-nH} dB_{s_1}^{j_1}\cdots dB_{s_n}^{j_n} dB_{s_{n+1}}^i .\end{aligned}$$

Using these results we can prove the following proposition.

Proposition 11 *The process $Y=\{Y_t, t\ge 0\}$ defined in (12) is not a fractional Brownian motion.*

Proof. If $Y_t=\sum_{i=1}^d\int_0^t f_i(B_s)dB_s^i$ is a fractional Brownian motion, then as in previous section one can show that

$$Z_t=\sum_{i=1}^d\int_0^t \eta_H(t,s)f_i(B_s)dB_s^i$$

is a classical Brownian motion. But

$$Z_t=\sum_{i=1}^d\sum_{n=1}^{\infty}\sum_{1\le j_1,\dots,j_n\le d} b_{i,j_1,\dots,j_n}\int_{\{0<s_1,\dots,s_{n+1}<t\}} h(t,s_1,\dots,s_{n+1})dB_{s_1}^{j_1}\cdots dB_{s_n}^{j_n}dB_{s_{n+1}}^i ,$$

where

$$h(t,s_1,\dots,s_{n+1})=\eta(t,s_1\vee s_2\vee\cdots\vee s_{n+1})(s_1\vee s_2\vee\cdots\vee s_{n+1})^{-nH} .$$

In a similar way to one dimensional case, one can show that $\{Z_t, t\ge 0\}$ is not a martingale ■

Proposition 12 *Let $Y=\{Y_t, t\ge 0\}$ be the process defined in (12) If $H\ge 2/3$, then Y_t is long range dependent and if $1/2<H<2/3$, then Y_t is not long range dependent.*

Proof. Set

$$\rho_n=\mathbb{E}\left[\left(\sum_{i=1}^d\int_0^1\frac{B_s^i}{|B_s|}\delta B_s^i\right)\left(\sum_{i=1}^d\int_n^{n+1}\frac{B_s^i}{|B_s|}\delta B_s^i\right)\right].$$

By the formula for the expectation of the product of two divergence integrals we can write

$$
\begin{aligned}
\rho_n &= \sum_{i,j=1}^{d} \alpha_H \int_0^1 \int_n^{n+1} \mathbb{E}\left(\frac{B_s^i B_t^i}{|B_s||B_t|} \right) \; |t-s|^{2H-2} ds dt \\
&\quad + \sum_{i,j=1}^{d} \alpha_H^2 \int_0^1 \int_n^{n+1} \int_0^t \int_0^s \mathbb{E}\left(D_\theta^j \left(\frac{B_s^i}{|B_s|} \right) D_\sigma^i \left(\frac{B_t^j}{|B_t|} \right) \right) \\
&\quad \times |\theta - t|^{2H-2} |\sigma - s|^{2H-2} d\theta d\sigma ds dt \\
:&= \rho_n^1 + \rho_n^2.
\end{aligned}
$$

In order to estimate the term ρ_n^1 we make use of the orthogonal decomposition

$$B_t = \frac{R(t,s)}{s^{2H}} B_s + \beta_{s,t} Y,$$

where

$$\beta_{s,t}^2 = \frac{(st)^{2H} - R(t,s)^2}{s^{2H}},$$

and Y is a d-dimensional standard normal random variable independent of B_s . Set

$$\lambda_{st} = \frac{R(t,s)}{\beta_{s,t} s^{2H}} = \frac{\frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)}{s^H \left[(st)^{2H} - \frac{1}{4}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)^2 \right]^{1/2}}$$

As t tends to infinity and s belongs to $(0,1)$, the term λ_{st} behaves as $Hs^{-2H} t^{H-1}$. Hence, by Lemma 13

$$
\begin{aligned}
\mathbb{E}\left(\frac{\langle B_s, B_t \rangle}{|B_s||B_t|} \right) &= \mathbb{E}\left(\frac{\langle B_s, \lambda_{st} B_s + Y \rangle}{|B_s||\lambda_{st} B_s + Y|} \right) \\
&= \mathbb{E}\left(\frac{\langle B_1, s^H \lambda_{st} B_1 + Y \rangle}{|B_1||s^H \lambda_{st} B_1 + Y|} \right) \\
&= s^H \lambda_{st} \mathbb{E}\left(\frac{|B_1|^2 |Y|^2 - \langle B_1, Y \rangle^2}{|B_1| \, |Y|^3} \right) + o(t^{H-1}).
\end{aligned}
$$

Hence,

$$\mathbb{E}\left(\frac{\langle B_s, B_t \rangle}{|B_s||B_t|} \right) \approx HCs^{-H} t^{H-1},$$

where

$$C=\mathbb{E}\left(\frac{|B_1|^2\,|Y|^2-\langle B_1,Y\rangle^2}{|B_1|\,|Y|^3}\right)>0.$$

This implies that the term ρ_n^1 behaves as n^{3H-3}.

For the term ρ_n^2 we have

$$\begin{aligned}\rho_n^2 &= H^2\sum_{i,j=1}^d\int_0^1\int_n^{n+1}\mathbb{E}\left(\left(\frac{\delta_{ij}}{|B_s|}-\frac{B_s^iB_s^j}{|B_s|^3}\right)\left(\frac{\delta_{ij}}{|B_t|}-\frac{B_t^iB_t^j}{|B_t|^3}\right)\right)\\
&\quad\times(s^{2H-1}+(t-s)^{2H-1})(t^{2H-1}-(t-s)^{2H-1})dsdt\\
&= H^2\int_0^1\int_n^{n+1}\mathbb{E}\left(\frac{d}{|B_s||B_t|}-\frac{|B_t|^2}{|B_s||B_t|^3}-\frac{|B_s|^2}{|B_s|^3|B_t|}+\frac{\langle B_s,B_t\rangle^2}{|B_s|^3|B_t|^3}\right)\\
&\quad\times(s^{2H-1}+(t-s)^{2H-1})(t^{2H-1}-(t-s)^{2H-1})dsdt\\
&= H^2\int_0^1\int_n^{n+1}\mathbb{E}\left(\frac{d-2}{|B_s||B_t|}+\frac{\langle B_s,B_t\rangle^2}{|B_s|^3|B_t|^3}\right)\\
&\quad\times(s^{2H-1}+(t-s)^{2H-1})(t^{2H-1}-(t-s)^{2H-1})dsdt.\end{aligned}$$

The term

$$\mathbb{E}\left(\frac{1}{|B_s||B_t|}\left(d-2+\frac{\langle B_s,B_t\rangle^2}{|B_s|^2|B_t|^2}\right)\right)$$

behaves as Kt^{-H} as t tends to infinity, where

$$K=\mathbb{E}\left(\frac{1}{|B_1||Y|}\left(d-2+\frac{\langle B_1,Y\rangle^2}{|B_1|^2|Y|^2}\right)\right)>0.$$

Hence, the term ρ_n^2 behaves also as n^{3H-3}. This completes the proof taking into account that the constants C and K are positive. ■

Lemma 13 *Let X and Y be independent d-dimensional standard normal random variables. Then as ε tends to zero we have*

$$\mathbb{E}\left(\frac{\langle X,\varepsilon X+Y\rangle}{|X|\,|\varepsilon X+Y|}\right)=\varepsilon\mathbb{E}\left(\frac{|X|^2\,|Y|^2-\langle X,Y\rangle^2}{|X|\,|Y|^3}\right)+o(\varepsilon).$$

Proof. We have

$$\begin{aligned}
\mathbb{E}\left(\frac{\langle X,\varepsilon X+Y\rangle}{|X|\,|\varepsilon X+Y|}\right) &= \mathbb{E}\left(\frac{\langle X,\varepsilon X+Y\rangle}{|X|\,|\varepsilon X+Y|}-\frac{\langle X,Y\rangle}{|X|\,|Y|}\right)\\
&= \mathbb{E}\left(\frac{\langle X,\varepsilon X+Y\rangle\,|Y|-\langle X,Y\rangle\,|\varepsilon X+Y|}{|X|\,|\varepsilon X+Y|\,|Y|}\right)\\
&= \mathbb{E}\left(\frac{\varepsilon\,|X|^2\,|Y|-\varepsilon\,\langle X,Y\rangle^2/\,|Y|\ +o(\varepsilon)}{|X|\,|\varepsilon X+Y|\,|Y|}\right),
\end{aligned}$$

and that yields the desired estimation. ■

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